The Lattice of Propositional Proof Systems

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Preliminary Notes

Proof systems form a lattice

Propositional Proof systems form a lattice under p-simulation. Namely, the infimum of two proof systems F and G is the proof system H where a proof of φ in H is a pair consisting of two proofs of φ, one in F and G each. More formally, let $F \sqcap G$ be the function

$$F \sqcap G := p \mapsto \begin{cases} \varphi & \text{if } p = (p_1, p_2) \text{ and } F(p_1) = G(p_2) = \varphi \\ 1 & \text{otherwise} \end{cases} .$$

Then it is easily seen that $F \sqcap G \leq_p F$ and $F \sqcap G \leq_p G$, and every proof system H with $H \leq_p F$ and $H \leq_p G$ is p-simulated by $F \sqcap G$, $H \leq_p F \sqcap G$.

Similarly, the supremum of F and G is the proof system where a proof is either a proof in F or a proof in G, together with a marker which of both it is, formally

$$F_0 \sqcup F_1 := p \mapsto \begin{cases} \varphi & \text{if } p = (i, p') \text{ for } i = 0, 1 \text{ and } F_i(p') = \varphi \\ 1 & \text{otherwise} \end{cases} .$$

It is again easily seen that $F \leq_p F \sqcup G$, $G \leq_p F \sqcup G$ and $F \sqcup G \leq_p H$ for each proof system H that p-simulates F and G.

Some examples and conjectures

From [4] we know that CP and constant-depth Frege proofs (F_d) are incompatible w.r.t. p-simulation. Hence the question arises whether the infimum and supremum of these proof systems could be p-equivalent to some natural proof systems.

Now resolution is a lower bound for $CP \sqcap F_d$, but it is properly weaker: The weak pigeonhole principle PHP^{2n}_n has quasipolynomial size F_d proofs, and its negation has polynomial size CP refutations. Hence it has quasipolynomial size proofs in $CP \sqcap F_d$, whereas resolution refutations of $\neg PHP_n^{2n}$ have to be of exponential size.

An upper bound for the $CP \sqcup F_d$ are constant depth proofs in PTK of [2]. We conjecture that this is not the least upper bound. To prove that, we have to find a family of tautologies having polynomial size, constant depth proofs in PTK, and give superpolynomial lower bounds for proofs of them in CP and F_d.

By the results of [3, 1], another pair of incomparable proof systems are dag-like Resolution (R) and tree-like Cutting Planes (CP_{tree}). Obvious upper and lower bounds for their infimum and supremum are

$$R_{tree} \leq_p R_{dag} \sqcap CP_{tree} \quad \text{and} \quad R_{dag} \sqcap CP_{tree} \leq_p CP_{dag} \ .$$

But we conjecture that both are not optimal, i.e.

- there are sets of clauses that have polynomial size dag-like resolution refutations as well as tree-like CP refutation, but a superpolynomial lower bound for tree-like resolution, and
- there are sets of clauses that have polynomial size dag-like CP refutations, but superpolynomial lower bounds for dag-like resolution as well as tree-like CP.

References

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