A long-standing open problem in propositional proof complexity that seems to be just beyond the reach of current techniques is to prove lower bounds for $AC^0[2]$-Frege proof systems, i.e., proof systems where the proof lines are constant-depth formulas with an additional parity connective.

This paper considers three relatively strong families of subsystems of $AC^0[2]$-Frege proof systems, for which exponential lower bounds on proof size are known. In order of increasing strength, the subsystems are: constant-depth Frege proof systems with parity axioms, and the tree-like and dag-like versions of systems introduced by Krajíček [1] called $PK_d^c(\oplus)$.

In a $PK_d^c(\oplus)$-proof, lines are disjunctions (cedents) in which all disjuncts have depth at most d, parity conntctives can only appear as the outermost connectives of disjuncts, and at most c disjuncts in a line can contain a parity connective.

The paper shows that tree-like $PK_{O(1)}^{O(1)}(\oplus)$ is quasi-polynomially but not polynomially equivalent to constant-depth Frege systems with parity axioms. It is also verified that the technique for separating parity axioms from parity connectives due to Impagliazzo and Segerlind [2] can be adapted to give a super-polynomial separation between dag-like $PK_{O(1)}^{O(1)}(\oplus)$ and $AC^0[2]$-Frege; the technique is inherently unable to prove separations that are larger than quasi-polynomial.

The paper also considers a proof system related to the system Res-Lin introduced by Itsykson and Sokolov [3]. It is shown that an extension of tree-like Res-Lin is polynomially simulated by a system related to dag-like $PK_{O(1)}^{O(1)}(\oplus)$, and an exponential lower bound for this system is obtained.

References

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