This paper proposes a new framework to investigate the complexity of propositional proof systems, starting from the observation that most of the known lower bounds are shown for sequences of tautologies given uniformly.

A $\Delta_0(\alpha)$-formula is a bounded formula in the language of Peano Arithmetic with an additional predicate symbol α. For such a formula φ, the well-known *Paris-Wilkie* translation produces a sequence of propositional formulas, where the n^{th} formula expresses that φ holds of the integers up to n.

For a proof system P, the set U_P is defined as the set of those $\Delta_0(\alpha)$-formulas φ whose Paris-Wilkie-translations have polynomial size proofs in P. If P is polynomially bounded, then U_P coincides with the set T of $\Delta_0(\alpha)$-formulas that are true in the integers, and similarly, if P polynomially simulates Q, then $U_Q \subseteq U_P$, but the converses of these statements do not necessarily hold.

First, some known separations [1, 2] and lower bounds [3, 4] for bounded-depth Frege systems are phrased in this framework. As first steps into the proposed research direction, two topics are then studied: the arithmetic complexity of the sets U_P, and logical properties of these sets.

It is easily seen that T is complete for the class Π^0_1 of co-c.e. sets. Here it is shown that for every proof system P, the set U_P is in the class Σ^0_2 in the arithmetical hierarchy, and hard for Π^0_1. Thus showing $U_P \notin \Pi^0_1$ for some P would imply super-polynomial lower bounds for P.

Finally, some closure properties of the sets U_P are shown under various assumptions on P. Moreover, the following result concerning a variant U'_P of U_P, defined w.r.t. a language extended by a function symbol for exponentiation, is obtained: Such a set U'_P is logically closed if and only if it coincides with the set T' of true $\Delta_0(\alpha)$-formulas in this language, in other words, for all proof systems P the set U'_P axiomatizes T'. The proof of this result uses an upper bound on the size of cut-free LK-proofs that might be of independent interest.

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